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Topological Quantization of Magnetic Monopoles and Their Bifurcation Theory

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Using $SU(2)$ gauge field theory and the ϕ-mapping method, we quantize the magnetic monopoles at the topological level and determine their quantum numbers by the Hopf indices and Brouwer degrees of the ϕ-mapping. Then, based on the implicit function theorem and Taylor expansion, we study the origin and bifurcation theories of magnetic monopoles at the limit points and bifurcation points (including first-order and second-order degenerate points), respectively. We point out that a magnetic monopole can split into at most four particles at one time.

1. INTRODUCTION

This paper continues our recent work on magnetic monopoles and topological current theory. In order to maintain the continuity of the whole work and make the background of this paper clear, in this section we give a brief review of our early work on the topological quantization of magnetic charges. In Section 2 we introduce the origin of the magnetic monopoles at the limit points. The bifurcation theories of magnetic monopoles at first-order and second-order degenerate points are investigated in Sections 3 and 4, respectively.

In previous papers (Duan and Ge, 1976, 1979; Duan, 1984; Duan and Liu, 1987), we have shown that the electromagnetic field is defined by 't Hooft (1974) in $SU(2)$ gauge field theory as

$$F_{\mu\nu} = F^a_{\mu\nu} n^a - \frac{1}{e} \varepsilon_{abc} n^a D_\mu n^b D_\nu n^c \tag{1}$$

$$a, b, c = 1, 2, 3; \qquad \mu, \nu = 0, 1, 2, 3$$

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where e is the electromagnetic coupling constant, $F^a_{\mu\nu}(x)$ stands for the gauge field tensor

$$F^a_{\mu\nu} = \partial_\mu A^a_\mu - \partial_\nu A^a_\mu + e\varepsilon_{abc} A^b_\mu A^c_\nu \tag{2}$$

and $D_\mu n^a(x)$ is the covariant derivative of $n^a(x)$

$$D_\mu n^a = \partial_\mu n^a + e\varepsilon_{abc} A^b_\mu n^c \tag{3}$$

in which $A^a_\mu(x)$ is the $SU(2)$ gauge potential and $n^a(x)$ a unit vector field in isospace

$$n^a(x) n^a(x) = 1 \tag{4}$$

which can, in general, be further expressed by

$$n^a(x) = \frac{\phi^a(x)}{\|\phi(x)\|}, \qquad \|\phi(x)\| = \sqrt{\phi^a(x)\phi^a(x)} \tag{5}$$

Here the fundamental field $\phi^a(x)$ in 't Hooft's theory is identified with the three-dimensional Higgs field ('t Hooft, 1976a,b).

The magnetic charge current j^μ_m is determined by the first pair of Maxwell equations

$$\partial_\nu F^{\mu\nu} = -4\pi j^\mu_m, \qquad F^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\lambda\rho} F_{\lambda\rho} \tag{6}$$

and it can be found that

$$j^\mu_m = \frac{1}{4\pi}\frac{1}{2}\frac{1}{e}\varepsilon^{\mu\nu\lambda\rho}\varepsilon_{abc}\partial_\nu n^a \partial_\lambda n^b \partial_\rho n^c \tag{7}$$

which is identically conserved, i.e.,

$$\partial_\mu j^\mu_m = 0 \tag{8}$$

Using the so-called ϕ-mapping method (e.g., Duan *et al.*, 1994) and (5), we obtain the δ-function-like current

$$j^\mu_m = \frac{1}{e}\delta(\vec{\phi}) J^\mu\left(\frac{\phi}{x}\right) \tag{9}$$

where the Jacobian determinants $J^\mu(\phi/x)$ are defined by

$$\varepsilon^{abc} J^\mu\left(\frac{\phi}{x}\right) = \varepsilon^{\mu\nu\lambda\rho}\partial_\nu\phi^a\partial_\lambda\phi^b\partial_\rho\phi^c \tag{10}$$

in which the component $J^0(\phi/x)$ is just the usual 3-dimensional Jacobian determinant of $\vec{\phi}(x)$ with respect to $\vec{x} = (x^1, x^2, x^3)$,

$$J^0\left(\frac{\phi}{x}\right) = J\left(\frac{\phi}{x}\right) = \frac{\partial(\phi^1, \phi^2, \phi^3)}{\partial(x^1, x^2, x^3)} \tag{11}$$

From (9), it is obvious that $j_m^\mu \neq 0$ only when $\vec{\phi}(x) = 0$.

Suppose that the fundamental field $\phi^a(x)$ possesses l isolated zeros. According to the implicit function theorem (Goursat, 1904), when the Jacobian determinant is given as

$$J\left(\frac{\phi}{x}\right) \neq 0 \tag{12}$$

the zeros of $\phi^a(x)$ can be expressed as the functions of the time like variable $x^0 = t$,

$$\vec{x} = \vec{z}_i(t), \qquad x^0 = z_i^0(t) = t, \qquad i = 1, \ldots, l \tag{13}$$

with the generalized velocities

$$\frac{dz_i^\mu}{dt} = \frac{J^\mu(\phi/x)}{J(\phi/x)}\bigg|_{z_i(t)}, \qquad \frac{dz_i^0}{dt} = 1 \tag{14}$$

On the other hand, as we proved in Duan *et al.* (1997), the δ-function $\delta(\vec{\phi})$ can be expanded by these zeros as

$$\delta(\vec{\phi}) = \sum_{i=1}^{l} \frac{\beta_i}{|J(\phi/x)_{z_i(t)}|}\delta(\vec{x} - \vec{z}_i(t)) \tag{15}$$

where the positive integer β_i is called the Hopf index of the ϕ-mapping at $z_i(t)$ (e.g., Duan and Meng, 1993) and it means that, when the point $\vec{x}$ covers the neighborhood of $\vec{z}_i(t)$ once, the function $\vec{\phi}(x)$ covers the corresponding region β_i times, which is a topological number of first Chern class and relates to the generalized winding number of the ϕ-mapping (see also Duan *et al.*, 1997). Then, substituting (14) and (15) into (9), we get the dynamic form of the magnetic charge current j_m^μ,

$$j_m^\mu = \frac{1}{e}\sum_{i=1}^{l} \beta_i\eta_i\delta(\vec{x} - \vec{z}_i(t))\frac{dz_t^\mu}{dt} \tag{16}$$

and the topological quantization of the magnetic monopole desntiy ρ_m

$$\rho_m = j_m^0 = \frac{1}{e}\sum_{i=1}^{l} \beta_i\eta_i\delta(\vec{x} - \vec{z}_i(t)) \tag{17}$$

where

$$\eta_i = sign J\left(\frac{\phi}{x}\right)\Bigg|_{z_i(t)} = \pm 1 \tag{18}$$

is called the Brouwer degree of the ϕ-mapping at $z_i(t)$ (Duan and Meng, 1993).

From the above discussions, we see that (i) the zeros of the Higgs field $\vec{\phi}(x)$ are just the sources of the magnetic monopoles, and the motions of the magnetic monopoles are the same as the equations of $\vec{\phi}$'s zeros, (ii) the magnetic monopoles are topologically quantized in the unit of the basic magnetic charge $g_0 = 1/e$ and the topological quantum numbers are determined by the Hopf indices β_i and Brouwer degrees η_i of the ϕ-mapping at its zeros, and (iii) the Brouwer degree $\eta_i = +1$ corresponds to the magnetic monopole, while $\eta_i = -1$ corresponds to the anti-magetic monopole.

2. THE ORIGIN OF MAGNETIC MONOPOLES

In the previous section, we achieved the topological quantization of magnetic monopoles and obtained the dynamic form of magnetic charge current under the condition (12), which guarantees that all of the zeros of the Higgs field are regular points of the ϕ-mapping. However, when the kernel of the ϕ-mapping contains some branch points at which (12) fails, the above results change. Now, let us explore what happens to the magnetic monopoles at the branch point $x^* = (t^*, \vec{x}^*)$ determined by

$$\begin{cases} \phi^1(t, x^1, x^2, x^3) = 0 \\ \phi^2(t, x^1, x^2, x^3) = 0 \\ \phi^3(t, x^1, x^2, x^3) = 0 \\ \phi^4(t, x^1, x^2, x^3) = J(\phi/x) = 0 \end{cases} \tag{19}$$

In ϕ-mapping theory, these are usually two kinds of branch points, the limit points and bifurcation points (Kubicek and Marek, 1983), satisfying

$$J^i\left(\frac{\phi}{x}\right)\Bigg|_{x^*} \neq 0, \qquad i = 1, 2, 3 \tag{20}$$

and

$$J^i\left(\frac{\phi}{x}\right)\Bigg|_{x^*} = 0, \qquad i = 1, 2, 3 \tag{21}$$

respectively, where the Jacobian determinants $J^i(\phi/x)$ have been defined generally in (10). Here, we consider the case (20). The other case (21) is complicated and will be treated in Sections 3 and 4. For simplicity and without lose of generality, we chose $i = 1$.

It is well known that the usual implicit function theorem is of no use when the Jacobian determinant $J(\phi/x) = 0$. So we use the Jacobian determinant $J^1(\phi/x)$ instead of $J(\phi/x)$ to search for the solution of the equation $\bar{\phi}(x) = 0$. This means we will replace the timelike variable $x^0 = t$ by x^1. To see this point clearly, we rewrite the first three equations of (19) as

$$\bar{\phi}(x^1, t, x^2, x^3) = 0 \tag{22}$$

Considering the condition (20) and making use of the implicit function theorem, we can express the solution of (22) in the neighborhood of the limit point $x^* = (t^*, \bar{x}^*)$ as

$$t = t(x^1), \qquad x^2 = x^2(x^1), \qquad x^3 = x^3(x^1) \tag{23}$$

with $t^* = t(x^{1*})$. In order to show the behavior of magnetic monopoles at the limit point, let us investigate the Taylor expansion of (23) in the neighborhood of $x^* = (t^*, \bar{x}^*)$,

$$t = t^* + \left.\frac{dt}{dx^1}\right|_{x^*}(x^1 - x^{1*}) + \frac{1}{2}\left.\frac{d^2t}{(dx^1)^2}\right|_{x^*}(x^1 - x^{1*})^2 \tag{24}$$

From (14), (20), and the last equation of (19), one has

$$\frac{dx^1}{dt} = \left.\frac{J^1(\phi/x)}{J(\phi/x)}\right|_{x^*} = \infty \tag{25}$$

that is,

$$\left.\frac{dt}{dx^1}\right|_{x^*} = 0 \tag{26}$$

Then the expansion (24) is further represented by

$$t - t^* = \frac{1}{2}\left.\frac{d^2t}{(dx^1)^2}\right|_{x^*}(x^1 - x^{1*})^2 \tag{27}$$

which is a parabola in the x^1 versus t plane. From (27) we can obtain two solutions, $x_1^1(t)$ and $x_2^1(t)$, which give the branch solutions of magnetic monopoles at the limit point. If $d^2t/(dx^1)^2|_{x^*} > 0$, we have the branch solutions for $t > t^*$. If $d^2t/(dx^1)^2|_{x^*} < 0$, we have the branch solutions for $t < t^*$. The former case is related to the origin of magnetic monopoles at the limit point. Since the magnetic charge current is identically conserved, the topological quantum numbers of these two generated magnetic monopoles must be opposite, i.e.,

$$\beta_1\eta_1 = -\beta_2\eta_2 \tag{28}$$

or

$$\beta_1 = \beta_2, \qquad \eta_1 = -\eta_2 \tag{29}$$

which is important in the early universe because of spontaneous symmetry breaking.

3. THE BIFURCATION OF MAGNETIC MONOPOLES

Now let us consider the other case (21). In the present condition, we have the restrictions

$$J\left(\frac{\phi}{x}\right)\Bigg|_{x^*} = 0, \qquad J^i\left(\frac{\phi}{x}\right)\Bigg|_{x^*} = 0, \qquad i = 1, 2, 3 \tag{30}$$

i.e., the rank of the Jacobian matrix $[\partial\phi/\partial x]$ is given by

$$rank\left[\frac{\partial\phi}{\partial x}\right]\Bigg|_{x^*} < 3 \tag{31}$$

The two restrictive conditions in (30) imply the important fact that the function relationship between t and $\vec{x}$ is not unique in the neighborhood of the bifurcation point x^*. In our dynamic form of magnetic charge current, this fact can be seen easily from equation (14),

$$\frac{dx^i}{dt} = \frac{J^i(\phi/x)}{J(\phi/x)}\Bigg|_{x^*}, \qquad i = 1, 2, 3 \tag{32}$$

which under the condition (30) directly shows the indefiniteness of the direction of the integral curve of (32) at x^*. This is why the very point $x^* = (t^*, \vec{x}^*)$ is called the bifurcation point of magnetic charge current.

Since the rank of the Jacobian matrix $[\partial\phi/\partial x]$ is less than 3, we suppose

$$rank\left[\frac{\partial\phi}{\partial x}\right]\Bigg|_{x^*} = 3 - 1 = 2 \tag{33}$$

and let

$$J_1\left(\frac{\phi}{x}\right)\Bigg|_{x^*} = \begin{vmatrix}\partial\phi^1/\partial x^2, & \partial\phi^1/\partial x^3\\ \partial\phi^2/\partial x^2, & \partial\phi^2/\partial x^3\end{vmatrix}_{x^*} \neq 0 \tag{34}$$

which means x^* is a first-order degenerate point of the ϕ-mapping. (The case that x^* is a second-order degenerate point will be detailed in the next section.) From $\phi^1(x) = 0$ and $\phi^2(x) = 0$, the implicit function theorem says that there exists one and only one system of function relationships

$$x^2 = x^2(t, x^1), \qquad x^3 = x^3(t, x^1) \tag{35}$$

Substituting (35) into ϕ^1 and ϕ^2, we get

$$\phi^b(t, x^1, x^2(t, x^1), x^3(t, x^1)) \equiv 0, \qquad b = 1, 2 \tag{36}$$

which give

$$\sum_{j=2}^{3} \phi^b_j x^j_0 = -\phi^b_0, \qquad \sum_{j=2}^{3} \phi^b_j x^j_1 = -\phi^b_1 \tag{37}$$

$$\sum_{j=2}^{3} \phi^b_j x^j_{00} = -\sum_{j=2}^{3} [2\phi^b_{j0} x^j_0 + \sum_{k=2}^{3} (\phi^b_{jk} x^k_0) x^j_0] - \phi^b_{00} \tag{38}$$

$$\sum_{j=2}^{3} \phi^b_j x^j_{01} = -\sum_{j=2}^{3} [\phi^b_{j0} x^j_1 + \phi^b_{j1} x^j_0 + \sum_{k=2}^{3} (\phi^b_{jk} x^k_0) x^j_1] - \phi^b_{01} \tag{39}$$

$$\sum_{j=2}^{3} \phi^b_j x^j_{11} = -\sum_{j=2}^{3} [2\phi^b_{j1} x^j_1 + \sum_{k=2}^{3} (\phi^b_{jk} x^k_1) x^j_1] - \phi^b_{11} \tag{40}$$

where $b = 1, 2$; $j,k = 2, 3$; and

$$x^j_0 = \frac{\partial x^j}{\partial t}, \; x^j_1 = \frac{\partial x^j}{\partial x^1}, \; x^j_{00} = \frac{\partial^2 x^j}{\partial t^2}, \; x^j_{01} = \frac{\partial^2 x^j}{\partial t \partial x^1}, \; x^j_{11} = \frac{\partial^2 x^j}{(\partial x^1)^2} \tag{41}$$

$$\phi^b_0 = \frac{\partial \phi^b}{\partial t}, \; \phi^b_1 = \frac{\partial \phi^b}{\partial x^1}, \; \phi^b_j = \frac{\partial \phi^b}{\partial x^j}, \; \phi^b_{00} = \frac{\partial^2 \phi^b}{\partial t^2}, \; \phi^b_{01} = \frac{\partial^2 \phi^b}{\partial t \partial x^1} \tag{42}$$

$$\phi^b_{11} = \frac{\partial^2 \phi^b}{(\partial x^1)^2}, \; \phi^b_{j0} = \frac{\partial^2 \phi^b}{\partial t \partial x^j}, \; \phi^b_{j1} = \frac{\partial^2 \phi^b}{\partial x^1 \partial x^j}, \; \phi^b_{jk} = \frac{\partial^2 \phi^b}{\partial x^j \partial x^k} \tag{43}$$

From these expressions we can calculate the values of the first- and second-order partial derivatives of (35) with respect to t and x^1 at the bifurcation point x^*.

With the aim of finding the different directions of all branch curves at the bifurcation point, as before, let us study the Taylor expansion of

$$F(t, x^1) = \phi^3(t, x^1, x^2(t, x^1), x^3(t, x^1)) \tag{44}$$

in the neighborhood of x^*, which, according to equations (19), must vanish at the bifurcation point, i.e.,

$$F(t^*, x^{1^*}) = 0 \tag{45}$$

From (44), the first-order partial derivatives of $F(t, x^1)$ with respect to $(t, x^1$t and $(t, x^1$x^1 by

$$\frac{\partial F}{\partial t} = \frac{\partial \phi^3}{\partial t} + \sum_{j=2}^{3} \frac{\partial \phi^3}{\partial x^j} x_0^j, \qquad \frac{\partial F}{\partial x^1} = \frac{\partial \phi^3}{\partial x^1} + \sum_{j=2}^{3} \frac{\partial \phi^3}{\partial x^j} x_1^j \tag{46}$$

On the other hand, making use of (34), (37), (46), and Cramer's rule, it is not difficult to prove that the two restrictive conditions in (30) can be rewritten as

$$J\left(\frac{\phi}{x}\right)\bigg|_{x^*} = \left(\frac{\partial F}{\partial x^1} J_1\left(\frac{\phi}{x}\right)\right)\bigg|_{x^*} = 0 \tag{47}$$

$$J^1\left(\frac{\phi}{x}\right)\bigg|_{x^*} = \left(\frac{\partial F}{\partial t} J_1\left(\frac{\phi}{x}\right)\right)\bigg|_{x^*} = 0 \tag{48}$$

which lead to

$$\frac{\partial F}{\partial t}\bigg|_{x^*} = 0, \qquad \frac{\partial F}{\partial x^1}\bigg|_{x^*} = 0 \tag{49}$$

by considering (34). The second-order partial derivatives of the function $F(t, x^1)$ are easily found to be

$$\frac{\partial^2 F}{\partial t^2} = \phi_{00}^3 + \sum_{j=2}^{3} [2\phi_{j0}^3 x_0^j + \phi_j^3 x_{00}^j + \sum_{k=2}^{3} (\phi_{jk}^3 x_0^k) x_0^j] \tag{50}$$

$$\frac{\partial^2 F}{\partial t \partial x^1} = \phi_{11}^3 + \sum_{j=2}^{3} [\phi_{j0}^3 \mathrm{x}_1^j + \phi_{j1}^3 x_0^j + \phi_j^3 x_{01}^j + \sum_{k=2}^{3} (\phi_{jk}^3 x_0^k) x_1^j] \tag{51}$$

$$\frac{\partial^2 F}{(\partial x^1)^2} = \phi_{11}^3 + \sum_{j=2}^{3} [2\phi_{j1}^3 x_1^j + \phi_j^3 x_{11}^j + \sum_{k=2}^{3} (\phi_{jk}^3 x_1^k) x_1^j] \tag{52}$$

which at $x^* = (t^*, \bar{x}^*)$ are denoted by

$$A = \frac{\partial^2 F}{\partial t^2}\bigg|_{x^*}, \qquad B = \frac{\partial^2 F}{\partial t \partial x^1}\bigg|_{x^*}, \qquad C = \frac{\partial^2 F}{(\partial x^1)^2}\bigg|_{x^*} \tag{53}$$

where j, $k = 2$, 3 and

$$\phi_j^3 = \frac{\partial \phi^3}{\partial x^j}, \qquad \phi_{00}^3 = \frac{\partial^2 \phi^3}{\partial t^2}, \qquad \phi_{01}^3 = \frac{\partial^2 \phi^3}{\partial t \partial x^1}, \qquad \phi_{11}^3 = \frac{\partial^2 \phi^3}{(\partial x^1)^2} \tag{54}$$

$$\phi_{j0}^3 = \frac{\partial^2 \phi^3}{\partial t \partial x^j}, \qquad \phi_{j1}^3 = \frac{\partial^2 \phi^3}{\partial x^1 \partial x^j}, \qquad \phi_{jk}^3 = \frac{\partial^2 \phi^3}{\partial x^j \partial x^k} \tag{55}$$

So, from (45), (49), and (53), we obtain the Taylor expansion of $F(t, x^1)$,

$$F(t, x^1) = \frac{1}{2} A(t - t^*)^2 + B(t - t^*)(x^1 - x^{1*}) + \frac{1}{2} C(x^1 - x^{1*})^2 \tag{56}$$

which by (44) is the behavior of $\phi^3(x)$ in the neighborhood of the bifurcation point x^*. Because of the third equation of (19) and letting $F(t, x^1) = 0$, we have

$$A(t - t^*)^2 + 2B(t - t^*)\,(x^1 - x^{1*}) + C(x^1 - x^{1*})^2 = 0 \tag{57}$$

which is followed by

$$A\left(\frac{dt}{dx^1}\right)^2 + 2B\frac{dt}{dx^1} + C = 0 \tag{58}$$

or

$$C\left(\frac{dx^1}{dt}\right)^2 + 2B\frac{dx^1}{dt} + \mathrm{A} = 0 \tag{59}$$

The different directions of the branch curves at the bifurcation point are determined by (58) or (59). The remaining component can be deduced by

$$\frac{dx^j}{dt} = x_0^j + x_1^j \frac{dx^1}{dt}, \qquad j = 2, 3 \tag{60}$$

As before, since the topological charge current of magnetic monopoles is identically conserved, the sum of the topological quantum numbers of these two split magnetic monopoles must be equal to that of the original monopole at the bifurcation point, i.e.,

$$\beta_1\eta_1 + \beta_2\eta_2 = \beta\eta \tag{61}$$

We conclude that in our ϕ-mapping theory of magnetic charge current, there exists the crucial case of the branching process; when an original magnetic monopole moves through the bifurcation point, it may split into two magnetic monopoles moving along different branch curves.

4. THE BIFURCATION AT A SECOND-ORDER DEGENERATE POINT

In the preceding section we studied the bifurcation of a magnetic monopole at a first-order degenerate point. In this section, we discuss the branching process of the magnetic charge current at a second-order degenerate point $x^* = (t^*, \bar{x}^*)$, at which the rank of the Jacobian matrix $[\partial\phi/\partial x]$ is

$$rank \left[\frac{\partial \phi}{\partial x} \right] \Bigg|_{x^*} = 3 - 2 = 1 \tag{62}$$

Suppose that

$$\frac{\partial \phi^1}{\partial x^3} \Bigg|_{x^*} \neq 0 \tag{63}$$

With the same reasoning as in obtaining (35), from $\phi^1(x) = 0$ we have the function relationship

$$x^3 = x^3(t, x^1, x^2) \tag{64}$$

in the neighborhood of x^*. In order to determine the values of the first- and second-order partial derivatives of x^3 with respect to t, x^1, and x^2, one can derive easily a system of equations similar to (36)–(43). Substituting the relationship (64) into $\phi^2(x) = 0$ and $\phi^3(x) = 0$, we get

$$\begin{cases} F_1(t, x^1, x^2) = \phi^2(t, x^1, x^2, x^3(t, x^1, x^2)) = 0 \\ F_2(t, x^1, x^2) = \phi^3(t, x^1, x^2, x^3(t, x^1, x^2)) = 0 \end{cases} \tag{65}$$

As we showed in the previous section, for the first-order partial derivatives of the functions $F_1(t, x^1, x^2)$ and $F_2(t, x^1, x^2)$, we can prove the following six formulas similar to (49):

$$\frac{\partial F_c}{\partial t} \Bigg|_{x^*} = 0, \qquad \frac{\partial F_c}{\partial x^1} \Bigg|_{x^*} = \phi, \qquad \frac{\partial F_c}{\partial x^2} \Bigg|_{x^*} = \phi, \qquad c = 1, 2 \tag{66}$$

So the Taylor expansions of $F_1(t, x^1, x^2)$ and $F_2(t, x^1, x^2)$ can be written in the neighborhood of x^* by

$$\begin{aligned} F_c(t, x^1, x^2) \approx {} & A_{c1}(t - t^*)^2 + A_{c2}(t - t^*)(x^1 - x^{1*}) + A_{c3}(t - t^*)(x^2 - x^{2*}) \\ & + A_{c4}(x^1 - x^{1*})^2 + A_{c5}(x^1 - x^{1*})(x^2 - x^{2*}) + A_{c6}(x^2 - x^{2*})^2 = 0 \end{aligned} \tag{67}$$

where $c = 1, 2$ and

$$A_{c1} = \frac{1}{2}\frac{\partial^2 F_c}{\partial t^2}\bigg|_{x^*}, \qquad A_{c2} = \frac{\partial^2 F_c}{\partial t \partial x^1}\bigg|_{x^*}, \qquad A_{c3} = \frac{\partial^2 F_c}{\partial t \partial x^2}\bigg|_{x^*} \tag{68}$$

$$A_{c4} = \frac{1}{2}\frac{\partial^2 F_c}{(\partial x^1)^2}\bigg|_{x^*}, \qquad A_{c5} = \frac{\partial^2 F_c}{\partial x^1 \partial x^2}\bigg|_{x^*}, \qquad A_{c6} = \frac{1}{2}\frac{\partial^2 F_c}{(\partial x^2)^2}\bigg|_{x^*} \tag{69}$$

Dividing (67) by $(t - t^*)^2$ and taking the limit $t \to t^*$, one obtains the two quadratic equations of dx^1/dt and dx^2/dt,

$$A_{c1} + A_{c2}\frac{dx^1}{dt} + A_{c3}\frac{dx^2}{dt} + A_{c4}\left(\frac{dx^1}{dt}\right)^2 + A_{c5}\frac{dx^1}{dt}\frac{dx^2}{dt} + A_{c6}\left(\frac{dx^2}{dt}\right)^2 = 0 \tag{70}$$

and further, eliminating the variable dx^1/dt, one has the equation of dx^2/dt in the form of a determinant

$$\begin{vmatrix} A_{14} & A_{15}v + A_{12} & A_{16}v^2 + A_{13}v + A_{11} & 0 \\ 0 & A_{14} & A_{15}v + A_{12} & A_{16}v^2 + A_{13}v + A_{11} \\ A_{24} & A_{25}v + A_{22} & A_{26}v^2 + A_{23}v + A_{21} & 0 \\ 0 & A_{24} & A_{25}v + A_{22} & A_{26}v^2 + A_{23}v + A_{21} \end{vmatrix} = 0 \tag{71}$$

with the variable $v = dx^2/dt$, which is a fourth-order equation of dx^2/dt,

$$a_1\left(\frac{dx^2}{dt}\right)^4 + a_2\left(\frac{dx^2}{dt}\right)^3 + a_3\left(\frac{dx^2}{dt}\right)^2 + a_4\left(\frac{dx^2}{dt}\right) + a_5 = 0 \tag{72}$$

Therefore we get different directions of the branch curves at the second-order degenerate point x^*. The largest number of different branch curves is four, which means an original magnetic monopole with the topological quantum number $\beta\eta$ can split into at most four particles at one time with magnetic charges $\beta_l\eta_l$ $(l = 1, 2, 3, 4)$ satisfying

$$\beta_1\eta_1 + \beta_2\eta_2 + \beta_3\eta_3 + \beta_4\eta_4 = \beta\eta \tag{73}$$

5. CONCLUSIONS

Based on the 't Hooft $SU(2)$ gauge field theory, we achieved the topological quantization, origin, and bifurcation of magnetic monopoles. The zeros of the Higgs field are the sources of the magnetic monopoles and they are quantized at the topological level in units of the basic magnetic charge $g_0 = 1/e$. The topological quantum numbers are determined by the Hopf indices

β_i and Brouwer degrees η_i of the ϕ-mapping at its zeros, in which β_i is a topoligcal number of first Chern class and $\eta_i = +1$ stands for the magnetic monopole, while $\eta_i = -1$ stands for the anti-magnetic monopole. It is also shown that there exists the crucial case of branching in our ϕ-mapping theory of magnetic charge current. At the limit point of the ϕ-mapping the branching process corresponds to the origin of magnetic monopoles, and at the bifurcation point the different directions of all branching curves are calculated. The largest number of different branching curves is four, i.e., an original magnetic monopole can split into at most four particles at one time. Since the magnetic charge current is identically conserved, the sum of the magnetic charges of these generated or split magnetic monopoles must be equal to zero or the topological quantum number of the original magnetic monopole. This result is important in the early universe because of spontaneous symmetry breaking. We see that the branching process of magnetic monopoles is not a gradual change, but starts at a critical value of the arguments, i.e., it represents a sudden change.

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